10/10 Recitation Solutions

1. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$

   (a) Show that $A\mathbf{x} = \mathbf{b}$ has no solution.

   The RREF of $\begin{bmatrix} A & | & \mathbf{b} \end{bmatrix}$ is $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$. Since there is a pivot in the final column, there is no solution.

   (b) Find a least squares solution $\hat{\mathbf{x}}$ to $A\mathbf{x} = \mathbf{b}$.

   We must solve $A^TA\mathbf{x} = A^T\mathbf{b}$. Since $A^TA = \begin{bmatrix} 6 & -8 \\ -8 & 18 \end{bmatrix}$ and $A^T\mathbf{b} = \begin{bmatrix} -11 \\ 27 \end{bmatrix}$, we need to reduce the matrix $\left[\begin{array}{cc|c} 6 & -8 & -11 \\ -8 & 18 & 2 \end{array}\right]$. The RREF is $\left[\begin{array}{cc|c} 1 & 0 & \frac{9}{22} \\ 0 & 1 & \frac{37}{22} \end{array}\right]$.
   Hence $\hat{\mathbf{x}} = \begin{bmatrix} \frac{9}{22} \\ \frac{37}{22} \end{bmatrix}$.

   (c) How does $A\hat{\mathbf{x}}$ compare to $\mathbf{b}$, geometrically?

   $A\hat{\mathbf{x}}$ is the vector in $C(A)$ that is closest to $\mathbf{b}$ (i.e. $A\hat{\mathbf{x}} = \text{proj}_{C(A)}\mathbf{b}$)

2. Consider the points $(-1, 2), (0, 0), (1, -3), (2, 5)$ in the $xy-$plane.

   (a) Sketch the points in the $xy-$plane. Notice a parabola might better fit the data than a line; sketch one.

   (b) Find the parabola $y = a + bx + cx^2$ that best fits these data points.
   We have the system (with no solution):

$$\begin{aligned} 2 &= a - b + c \\ 0 &= a \\ -3 &= a + b + c \\ 5 &= a + 2b + 4c \end{aligned}$$

   Hence $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ -3 \\ 5 \end{bmatrix}$.

Thus $A^TA = \begin{bmatrix} 4 & 2 & 6 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{bmatrix}$ and $A^T\mathbf{b} = \begin{bmatrix} 4 \\ 5 \\ 19 \end{bmatrix}$. Now the RREF of $\left[A^TA \;\middle|\; A^T\mathbf{b}\right]$ is $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{9}{5} \\ 0 & 1 & 0 & -\frac{19}{10} \\ 0 & 0 & 1 & \frac{5}{2} \end{array}\right]$. Hence $y = -\frac{9}{5} - \frac{19}{10}x + \frac{5}{2}x^2$.

(c) Determine the vertical distances between this parabola and the data points (i.e. find the error vector).

The error vector is $\mathbf{e} = \mathbf{b} - A\hat{\mathbf{x}} = \begin{bmatrix} 2 \\ 0 \\ -3 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} -9/5 \\ -19/10 \\ 5/2 \end{bmatrix} = \begin{bmatrix} -3/5 \\ 9/5 \\ -9/5 \\ 3/5 \end{bmatrix}$.

Hence the vertical distances between the data points and line are $3/5, 9/5, 9/5$, and $3/5$, respectively. Moreover, the first and third data points are below the parabola, while the second and fourth are above the parabola (which follows from the signs of the entries).

(d) Work through commands in the second section of the **Julia9LeastSquares** notebook to check your work above and to plot the parabola of best fit.

(e) Use Julia to find a line of best fit. Compare the error vector in this case to the error vector in the parabola case to determine which curve better fits the data. Recall that to calculated the length of a vector, you can use the "norm(v)" function in Julia.

A line of best fit is of the form $y = ax + b$. The related system is:

$$\begin{aligned} 2 &= -m + b \\ 0 &= b \\ -3 &= m + b \\ 5 &= 2m + b \end{aligned}$$

Let $A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ -3 \\ 5 \end{bmatrix}$. Since least squares solutions minimize the lengths of the error vectors, we should compare the length of the error vectors. The length of the error vector in the parabola case is approximately 2.68, while the length of the error vector in the line case is approximately 5.67. Hence the parabola gives a better approximation.

3. Recall that a system $A\mathbf{x} = \mathbf{b}$ does not necessarily have a unique least squares solution; this occurs if A does not have linearly independent columns. Come up with an

example of such a 2×2 matrix A and a vector $\mathbf{b} \notin C(A)$ and find a least squares solution.

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then $A^TA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $A^T\mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Now, to find a least squares solution, we need to solve $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. One solution is $\hat{\mathbf{x}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. More generally, the general solution is: $\hat{\mathbf{x}} = \begin{bmatrix} 1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.